First Order Response to a Sinusoidal Input

The response of a single time constant τ_1 to sine wave (A*sin(ω*t)) of amplitude A and frequency ω:

$$Y(s) = \frac{1}{(\tau_1 * s + 1)} * \frac{A*\omega}{(s^2 + \omega^2)}$$

First order system (single time constant) — Sinusoidal input (sine wave)

ϕ is a negative phase shift (phase Lag)

$$Y(t) = \frac{A}{\omega} * \left[\frac{\tau*\omega^2}{[1 + (\tau_1*\omega)^2]} * e^{-t/\tau} + \frac{\omega*\sin(\omega*t + \phi)}{[1 + (\tau_1*\omega)^2]^{1/2}} \right]$$

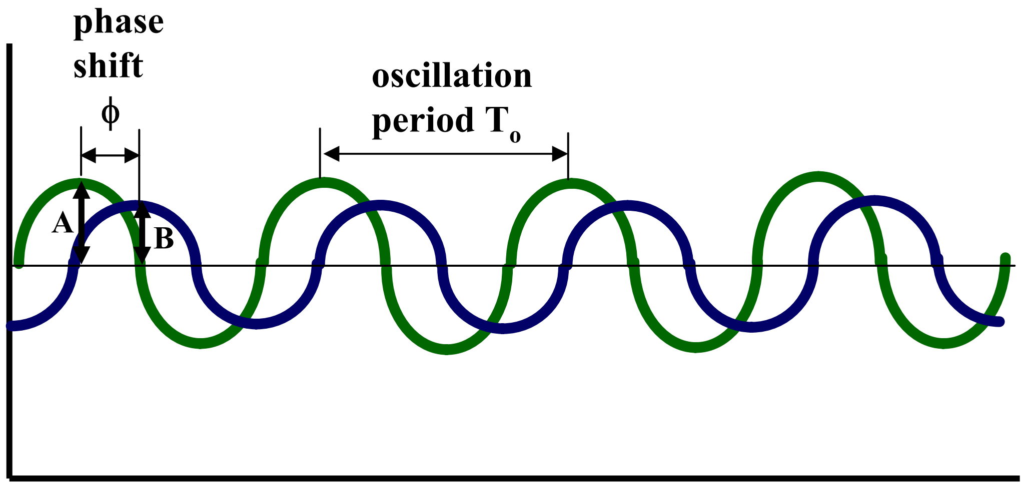
Transient response (decaying exponential) — Frequency response (sine wave)

The frequency response is the sine wave of constant amplitude B left after the transient has died out

$$B = \frac{A}{[1 + (\tau_1*\omega)^2]^{1/2}}$$

Amplitude Ratio and Phase Shift

If the phase shift is -180° between the process input and output, then the total shift for a control loop is -360° and the output is in phase with the input (resonance) since there is a -180° from negative feedback (control error = set point – process variable). This point sets the ultimate gain and period that is important for controller tuning.



$$AR = \frac{B}{A} = \frac{1}{[1 + (\tau_1 * \omega)^2]^{1/2}}$$ **amplitude ratio**

$\phi = -\mathrm{Tan}^{-1}(\omega * \tau)$ **negative phase shift**

(as ω approaches infinity, ϕ approaches -90° phase shift)

$\Delta t = (-360 / \phi) * T_o$ **time shift**

Amplitude ratios are multiplicative ($AR = AR_1 * AR_2$) and phase shifts are additive ($\phi = \phi_1 + \phi_2$)

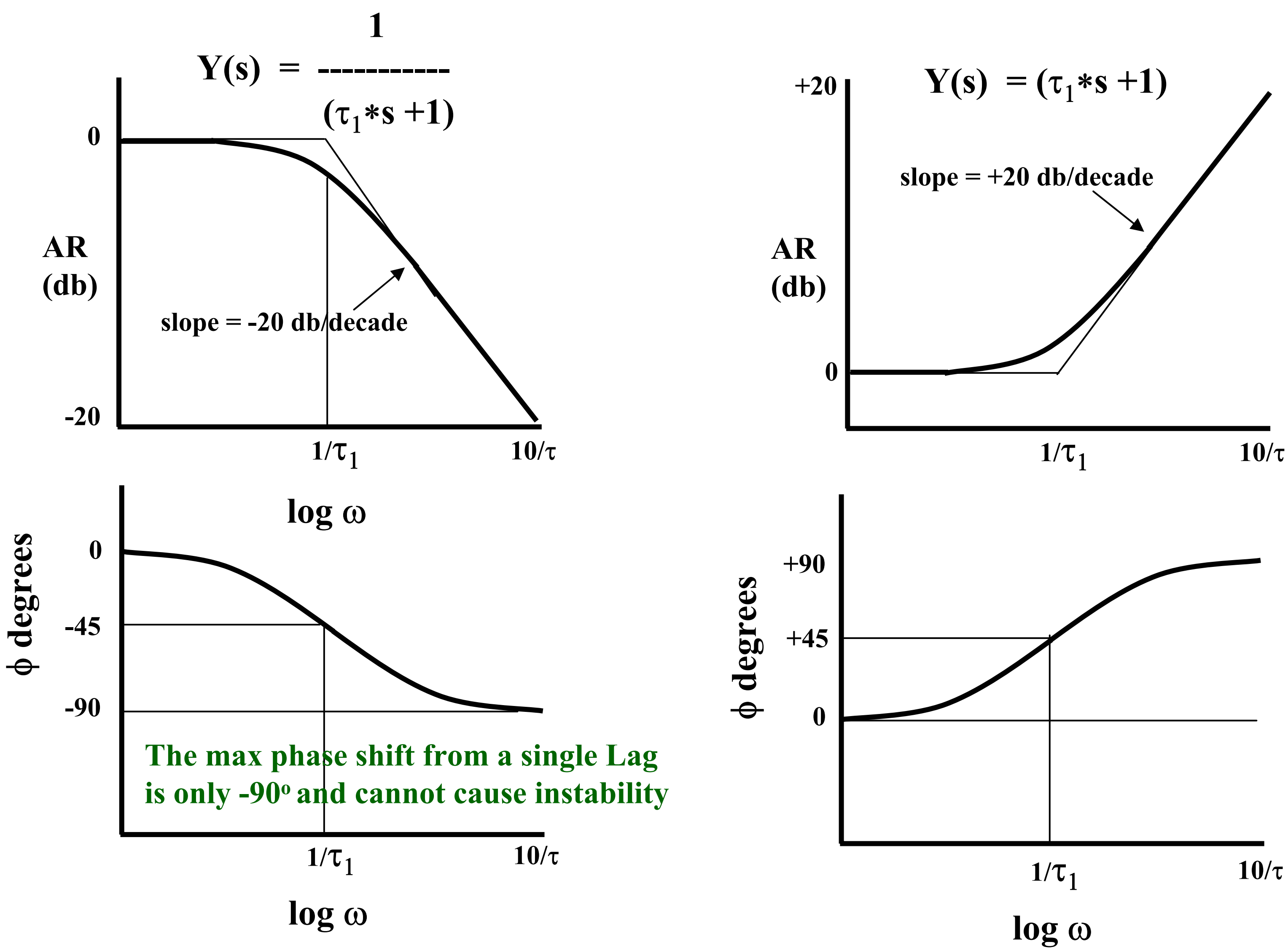
Basis of first order approx method where gains are multiplicative and dead times are additive

Bode Plots of Lag and Lead Time

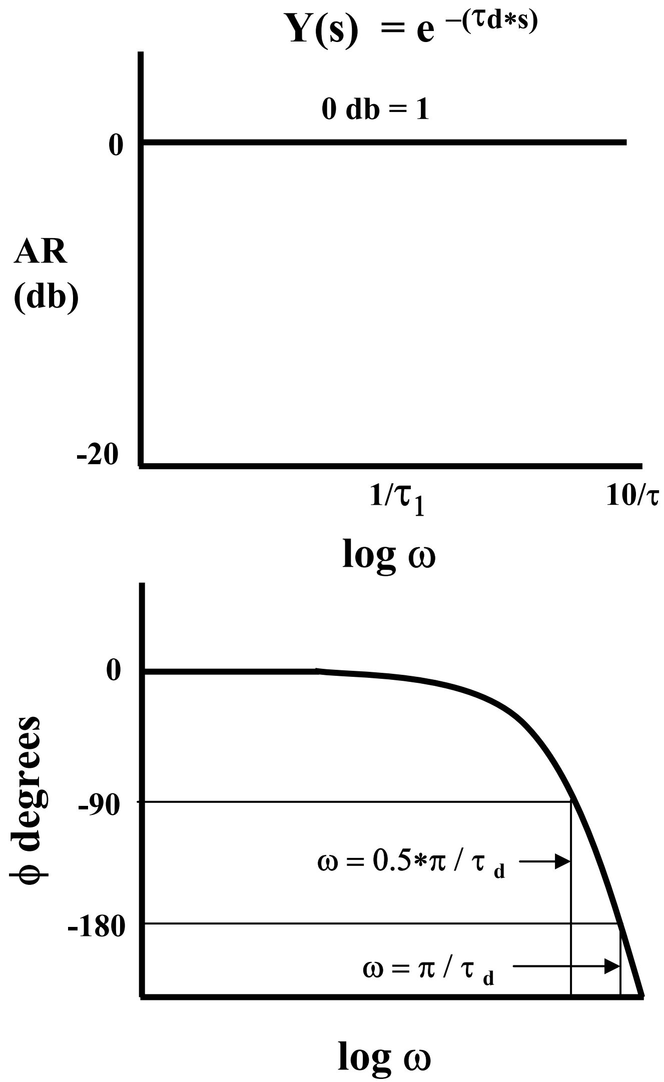
Lead is applied to feedforward signal
to compensate for a Lag in MV path

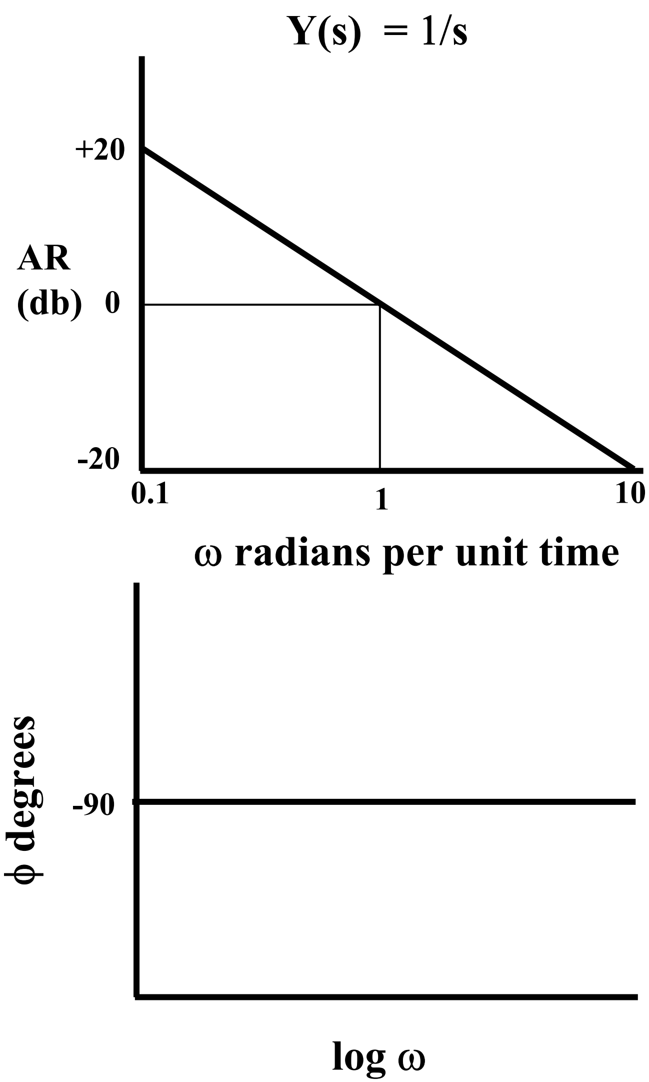
Lag Time

Lead Time



Bode Plots for Time Delay and Integrator

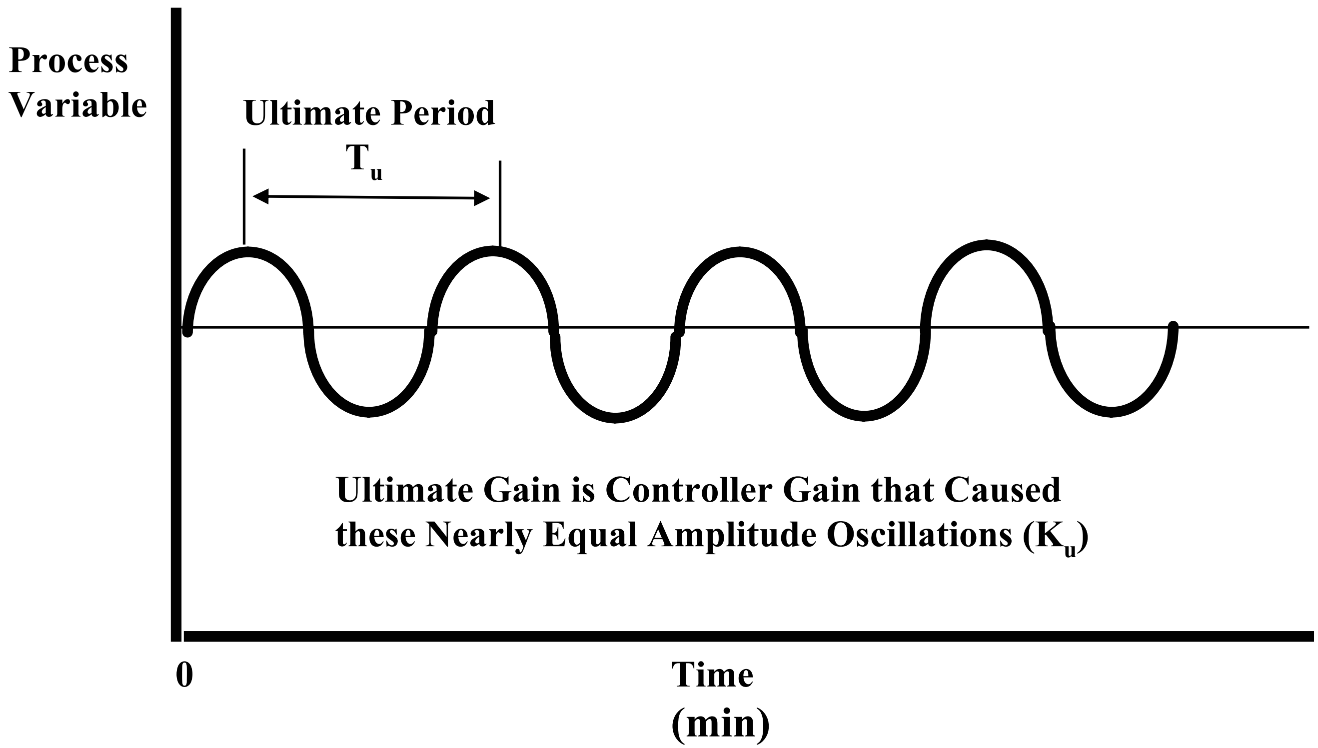




The phase shift from a time delay can be -180° and cause instability

The phase shift from a single integrator is only -90° and cannot cause instability

Ultimate Gain and Period



$$K_u = \frac{1}{K_o * AR_{-180}}$$

Ultimate gain of controller is inversely proportional to the product of the open loop static (process) gain and the amplitude ratio at -180° phase shift, which is the starting point of resonance and instability (growing oscillations)

$$T_u = \frac{2 * \pi}{\omega_n}$$

Ultimate period of control loop occurs at natural frequency ω_n (at -180 phase shift)

Modification of Ziegler Nichols (ZN) Closed Loop Method

For a Proportional plus Integral (PI) controller:

½ of ZN controller gain factor

$K_c = 0.2 * K_u$

$T_i = 1.0 * T_u$

For an integrating process to avoid slow nearly sustained oscillations:

10x ZN integral time factor

$T_i = 10 * T_u$

If clearly dead time dominant, dramatically increase the reset action

¼ of ZN integral time factor

$T_i = 0.2 * T_u$

For a Proportional plus Integral plus Derivative (PID) controller:

½ of ZN controller gain factor

$K_c = 0.3 * K_u$

$T_i = 0.5 * T_u$

$T_d = 0.1 * T_u$

Development of Ultimate Gain for a Self Regulating Process

$$K_u = \frac{1}{K_o * AR_{-180}}$$

substituting $$AR_{180} = \frac{1}{[1 + (\tau_1 * \omega_n)^2]^{1/2}}$$

$$K_u = \frac{[1 + (\tau_1 * \omega_n)^2]^{1/2}}{K_o}$$

substituting $$\omega_n = \frac{2 * \pi}{T_u}$$

$$K_u = \frac{[1 + (\tau_1 * 2 * \pi / T_u)^2]^{1/2}}{K_o}$$

Simplification of Ultimate Gain for a Self Regulating Process

$$K_u = \frac{[1 + (\tau_1 * 2 * \pi / T_u)^2]^{1/2}}{K_o}$$

For $T_u << \tau_1$ (loop dominated by a large time constant), the error is negligible for the following simplification:

$$K_u = \frac{2 * \pi * \tau_1}{K_o * T_u} = 1.5 * \frac{\tau_1}{K_o * \tau_d} \quad \text{since } T_u \cong 4 * \tau_d$$

For $T_u >> \tau_1$ (loop dominated by a large time delay), the error is negligible for the following simplification:

$$K_u = \frac{1}{K_o} \quad \text{(dead time dominant loop)}$$

Another way at arriving at the above equation is via the realization that the AR for a pure dead time is 1

nomenclature: τ_1= 1^{st} order time constant and τ_d= time delay

Development of Ultimate Gain for an Integrating Process

$$K_u = \frac{1}{K_o * AR_{-180}}$$

substituting $$AR_{180} = \frac{1}{\omega_n*[1 + (\tau_1*\omega_n)^2]^{1/2}}$$

$$K_u = \frac{\omega_n*[1 + (\tau_1*\omega_n)^2]^{1/2}}{K_o}$$

substituting $$\omega_n = \frac{2 * \pi}{T_u}$$

$$K_u = \frac{(2*\pi/T_u)*[1 + (\tau_1*2*\pi/T_u)^2]^{1/2}}{K_o}$$

Simplification of Ultimate Gain for an Integrating Process

$$K_u = \frac{(2*\pi/T_u)*[1 + (\tau_1*2*\pi/T_u)^2]^{1/2}}{K_o}$$

For $T_u << \tau_1$ (loop dominated by a large time constant), the error is negligible for the following simplification:

$$K_u = \frac{(2*\pi)^2*\tau_1}{K_o * T_u^2}$$

Note that dimensions of integrator K_o is 1/minute, which cancels out extra time unit in denominator

For $T_u >> \tau_1$ (loop dominated by a large time delay), the error is negligible for the following simplification:

$$K_u = \frac{2*\pi}{K_o * T_u}$$

(dead time dominant loop)

Another way at arriving at the above equation is via the realization that the AR for a pure dead time is 1

nomenclature: τ_1 = time constant and τ_d= time delay

Development of Ultimate Gain for a Runaway Process

$$K_u = \frac{1}{K_o * AR_{-180}}$$

$$\text{substituting } AR_{180} = \frac{1}{[1 + (\tau_1 * \omega_n)^2]^{1/2}\,[1 + (\tau_1' * \omega_n)^2]^{1/2}}$$

$$K_u = \frac{[1 + (\tau_1 * \omega_n)^2]^{1/2} * [1 + (\tau_1' * \omega_n)^2]^{1/2}}{K_o}$$

$$\text{substituting } \omega_n = \frac{2 * \pi}{T_u}$$

$$K_u = \frac{[1 + (\tau_1 * 2 * \pi / T_u)^2]^{1/2} * [1 + (\tau_1' * 2 * \pi / T_u)^2]^{1/2}}{K_o}$$

Simplification of Ultimate Gain for a Runaway Process

$$K_u = \frac{[1 + (\tau_1 * 2 * \pi / T_u)^2]^{1/2} * [1 + (\tau_1' * 2 * \pi / T_u)^2]^{1/2}}{K_o}$$

For $T_u << \tau_1$ (loop dominated by a large time constant), the error is negligible for the following simplification:

$$K_u = \frac{(2 * \pi)^2 * \tau_1 * \tau_1'}{K_o * T_u^2}$$

If $T_u >> \tau_1$ and $T_u >> \tau_1'$, then the loop is unstable for all controller settings because the ultimate gain is less than minimum controller gain based on the static gains (window of allowable gains is closed)

$$K_u = \frac{(2 * \pi) * \tau_1'}{K_o * T_u} < \frac{1}{K_o} \quad \text{(dead time dominant loop)}$$

Another way at arriving at the above equation is via the realization that the AR for a pure dead time is 1

τ_1 = negative feed back time constant, τ_d= time delay
τ_1' = positive feed back time constant, K_o = open loop gain

Development of Ultimate Period for a Self-Regulating Process

phase shift from time delay

$$\phi = -360 * f_n * \tau_d$$

substituting $f_n = 1/T_u$

$$T_u = (-360/\phi) * \tau_d$$

substituting $\phi = -180$ if all 180 of phase lag comes from time delay

$$T_u = (-360/-180) * \tau_d$$

$T_u = 2 * \tau_d$ for a process dominated by a time delay

substituting $\phi = -90$ if 90 of phase lag comes from a time constant so that 90 comes from time delay

Phase Lag implies a negative phase shift

$$T_u = (-360/-90) * \tau_d$$

$T_u = 4 * \tau_d$ for a process dominated by a time constant

Approximations for Ultimate Periods

(curve fits from Bode and Nyquist Plots)

For Self-Regulating Process:

$$T_u = 2 * \left[1 + \left[\frac{\tau_1}{\tau_1 + \tau_d} \right]^{0.65} \right] * \tau_d$$

For Integrating Process:

$$T_u = 4 * \left[1 + \left[\frac{\tau_1}{\tau_d} \right]^{0.65} \right] * \tau_d$$

For Runaway Process:

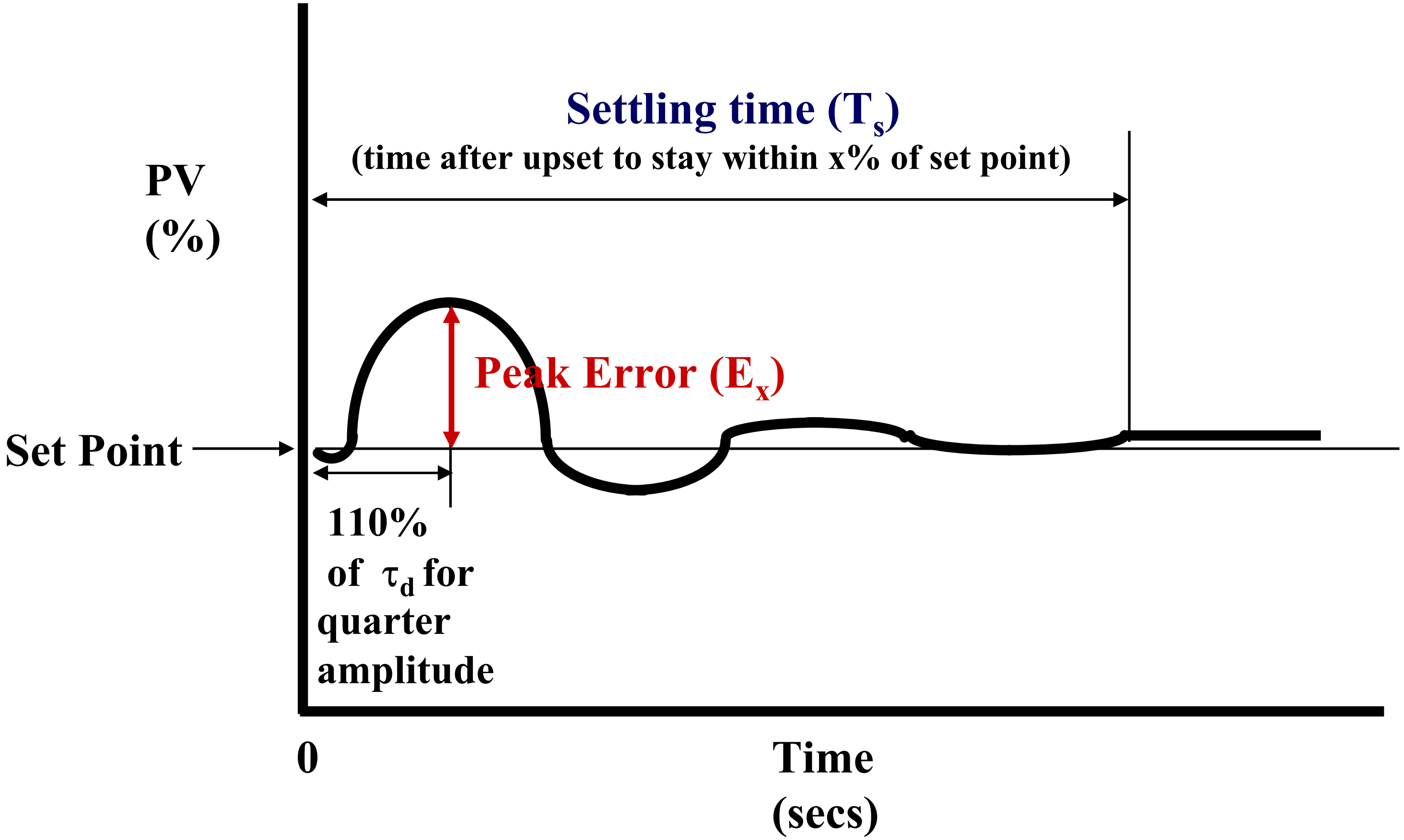
$$T_u = 4 * \left[1 + \left[\frac{N}{D} \right]^{0.65} \right] * \tau_d$$

$$N = (\tau_1' + \tau_1) * \tau_1' * \tau_1$$

$$D = (\tau_1' - \tau_1) * (\tau_1' - \tau_d) * \tau_d$$

As either τ_1 or τ_d approach τ_1', D approaches zero, T_u approaches infinity, and the controller is unstable for all tuning settings (window of allowable gains is closed)

Performance Criteria



Integrated Absolute Error (E_i) is integral of absolute magnitude of total error between the set point and the process variable. It corresponds to the total area between the process variable curve and the set point trajectory.

Development of Equation for Peak Error

Since a controller cannot either sense or compensate for an upset until after one dead time, the peak error is the exponential response after one dead time

$$E_x = [1 - e^{(-\tau_d / \tau_1)}] * E_o$$

For $\tau_d < \tau$

$$E_x = \frac{1}{1 + (\tau_d / \tau_1)} * E_o$$

$$E_x = \frac{\tau_d}{\tau_1 + \tau_d} * E_o$$

$$E_x = \frac{\tau_d}{\tau_1} * E_o$$

Using $K_c = \frac{\tau_1}{K_o * \tau_d}$

$$E_x = \frac{1}{K_o * K_c} * E_o$$